

WHITE PAPER

# The Development of Mathematical Thinking





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## Introduction

At DreamBox Learning we want to support teachers and parents in their commitment to their students' math education. So we've created this Math Development Timeline to help you understand the cognitive development and the appropriate math concepts children should be learning at each age from 4 to 8 years old. And we've suggested some appropriate ways that you can support math learning at each stage of development.

## The Transitional Nature of Development: A Cautionary Tale

Development is multifold in nature: it proceeds in patterns that are predictable enough that psychologists, doctors, and educators can point to broad social, physical, and mental behaviors that are typical of any age group. In addition, as the child grows each of these areas develop and impact each other; none develop in isolation.

Development is also unpredictable in that it is very individual, and patterns of growth can be unruly, at times aligning with the child's chronological age, or at other times, not. Parents and teachers know this from experience.

Although we will be describing each age separately as a way to organize the information, we will be describing the cognitive and mathematical development of children ages 4, 5, 6, 7 and 8 in broad strokes. There are no sharp divisions from one stage to another, but rather there are overlaps that indicate the transitional nature of growth.

## The Developing Mind from Ages 4–8

Young children already possess a rich assortment of mathematical cognitive abilities when they enter school. Through play with toys and everyday activities, they have spontaneously compared, sorted, arranged, and counted objects, explaining what they did and challenging others' explanations. (When their senses falter, they can always fall back on their fail-safe question, 'why?').

Young children are intensely curious about their environment and interact directly with it. Much of what they know is filtered through their perceptions, which are particular to them and can be very unreliable. Children at this intuitive stage will believe that a quantity changes when the arrangement is changed, even if they have counted several times. What's more, they don't question their belief.

As students' minds continue to develop, their brains go through successive stages of growth, and they become less dependent on perception. The quality of children's thinking changes; they think differently and are puzzled by their earlier belief that the quantity changed when the arrangement changed. Their more logical thinking tells them that no matter the arrangement, the quantity will remain the same.<sup>1</sup>

## Your Four Year Old

### COGNITIVE DEVELOPMENT

As any exhausted parent of a four year old can tell you, they are bursting with energy — capable of seemingly nonstop physical and mental activity. While four year olds' natural inquisitiveness leads to endless explorations, this has a downside: they tend to have short attention spans. Because four-year olds are developing "their large muscles, they need lots of physical play: running, jumping, climbing, and dancing. Four year olds learn best by moving large muscles."<sup>2</sup>

### MATH LEARNING AT AGE FOUR

In terms of mathematical development, four year olds may easily compare a set of objects to know which is more ("Hey, you have more grapes than I do!"), but have difficulty figuring out how many grapes they actually have. Although a four year old may recognize that the 'how many?' question means to count, they may struggle to do this.

Why? To count successfully a child needs to have coordinated some very powerful mathematical ideas. Spend some time observing your student, and you'll quickly realize that counting is not as easy or obvious as it seems. There are hurdles in successful counting: (1) each grape is counted once and only once; (2) there is a number name that corresponds to each grape counted; and (3) there is a logic to the counting sequence (base ten for the most part has a predictable pattern; early struggles with counting words may occur in the teen numbers and at the decades where the pattern breaks).

Four year olds are also fascinated with collecting and sorting objects and thinking

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about comparing themselves to others. There is often a mismatch in their use of language — for example the word 'bigger' may be used to describe different kinds of measurements (height, age, etc.)

### ACTIVITIES

Observing and listening while your student is engaged in play are the starting places for understanding his mathematical knowledge. If he counts, how does he count? Is it a sing-song or meaningful? Does he touch each object once? Is his voice in sync with his tag? Does he organize as he counts or does he not yet keep track of what's been counted?

Try to use puzzlement to help him develop counting strategies, saying, 'I wonder why we keep getting different numbers when we count? Does it matter? Is there something we could do to help us know what we've counted?' Try to resist showing your student what to do in that moment together. You surely don't want to rob her of the power (and beauty) of figuring it out on her own!

When you go outside, collect objects that attract his attention (stones, leaves, etc.). Then find different ways to sort this collection. Play a game like "Guess My Rule", where you sort the objects into groups with something in common like size or color, and your student guesses what the rule is (for example, everything in one group is red, and in the other group everything is small). Start a collection and ask, 'What might go with my things?' You might be surprised by his responses.

## Your Five Year Old

### COGNITIVE DEVELOPMENT

Inside the five year old lurks his four year old self, along with his three year self and so forth.<sup>3</sup> Thus the five year old is still an active explorer of the world around him, yet he is a calmer version of himself at four. A five year old is noticing more details, but continues to explain phenomena from what he intuits about them. He may be startled when confronting ideas that contradict his views. He may wonder if something is wrong with what he knows. This disequilibrium is important for broadening his cognitive horizons.

A five year old is thrilled with having wonderful ideas<sup>4</sup> but may look to adults to be reassured about her ideas. It is important to find ways to encourage the five year old without taking away her intellectual independence.

The climate of the classroom and social and learning interactions with friends are important in the learning process of the five year old. Even though he listens better and can hear how others have thought about finding a solution, he may still think there is only one way to do things — his! Most five year olds can stay with an activity for a longer time than they did when they were four. However, because they are still developing their fine motor skills, it is difficult for them to do extended work with paper and pencil. They learn best through active play and repetitive activities.

### MATH LEARNING AT AGE FIVE

The five year old is becoming less dependent on matching strategies to determine one-to-one correspondence, and more likely knows that for the five kids at his table he'll need five pencils. When

she counts, she can answer the question 'how many?' but may have a hazy idea that the last number counted means the total quantity. A more difficult question 'is it enough?' perplexes yet challenges her to find out. This kind of reasoning question can only be answered if a child understands how numbers are nested inside each other. A question like 'eight pencils and five children, are there enough for each child to have one pencil?' can only be understood if she knows that five is inside eight. Until a child has developed the idea that one set may be comprised of others, this will be a difficult question.

Although children at five are still developing the idea that the last number counted names the set, once they know it they are able to count on. This occurs naturally in the counting sequence. (There are six children at the birthday party; one more comes. Now there are seven.) Five year olds may easily know which set is more and they may be able to sequence quantities from smallest to largest. However, the question 'how much' more can present difficulties for five year olds. They will struggle to think about how much larger one quantity is than another.

## ACTIVITIES

You can transform everyday situations into opportunities for five year olds to think mathematically. If your student counts accurately you can help her to think about the permanence of a set of objects. Put out six pennies in a row and then change the arrangement. Will she think the quantity has changed? Playing games with quantities in different arrangements is one way for her to think about conservation of number — a big idea needed to understand addition and subtraction.

Some board and card games are appropriate, such as "Chutes and Ladders" and renaming the old game, "War", as "Who Has More?" where two players each turn over a card and the one who has more takes both cards. Then compare piles to see who has more.

Dominoes or dice games support the recognition of quantity without counting, and may lead to strategies such as counting on and learning basic double combinations.

Five year olds love repetition, so exploring patterns with your student is a natural way to develop his mathematical thinking. There are many examples of patterns in children's literature and song. You can use simple clapping patterns to help children discern sequences and predict what comes next. The ability to predict what comes next, based on being able to distinguish the unit in a pattern (for example, clap/ clap/snap might be the basic unit in a longer series of sounds) is an important tool in your child's mathematical toolbox.

## Your Six Year Old

### COGNITIVE DEVELOPMENT

Six is an age when children go through rapid physical, cognitive, and social change. At this age, there are major developmental shifts that affect how children view the world. Many six year olds are able to (1) organize concepts symbolically and systematically; (2) understand cause and effect; (3) understand a different point of view; and (4) reverse their thinking through mental activity. "The eagerness, curiosity, imagination, drive, and enthusiasm of the six year old are perhaps never again matched in quantity or intensity during the life span."<sup>5</sup>

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All facets of intellectual and physical growth are consolidating in the six year old, albeit unevenly. She is gaining fluency in verbally expressing her ideas and thoughts, but is still an active learner who likes to plunge into projects. Her organizational skills are being honed. She is beginning to be selective about which things should be attended to over others and can tune out distractions in the classroom.<sup>6</sup>

### MATH LEARNING AT AGE SIX

Mathematically, the six year old is developing a more complex understanding of number. One big idea is that numbers are nested inside each other (like Russian nesting dolls). With this critical piece of knowledge, he recognizes that 6 can be decomposed as 5 and 1, 3 and 3, etc., and he is firm in his belief that all sets of 6, no matter what objects, are equivalent.

These big ideas, along with his understanding of cardinality (the last number counted is the number of the set), underpin more efficient counting strategies such as counting on from the larger number, which is connected to knowing that if  $2 + 5 = 7$ , then  $5 + 2 = 7$ . Developing the idea of set is a critical stepping stone in a student's mathematical journey. Without this big idea, she will have difficulty understanding addition (the idea of two sets joining to create a new set). The six-year-old learner is also developing the idea that "nothing" can be represented by the numeral 0. This is a big idea that has relevance to the workings of the base-ten number system: that any number in the system can be written with the digits 0 to 9.

### ACTIVITIES

Board games, dice games, and card games are fun for the six year old and can help

her gain increasing fluency with addition combinations. Teachers and parents can adapt traditional card games, where the rules are familiar, to turn them into number games: The game adapted for five year olds as Who Has More? can challenge the six year old in two ways: with a new question, 'If you wanted them to be the same, how much more would you need?' Or, with each player turning over 2 cards and finding the sum, still asking 'who has more?'

One way to engage a child's thinking is to play a game, "Hidden Counters" in which part of a set is hidden. For example, count out eight pennies (making sure she knows there are eight). Hide four pennies under a cup, leaving the rest visible. Ask, 'How many are under the cup?' Notice her strategies for figuring this out. Does she know automatically that four are under the cup because she knows  $4 + 4 = 8$ ? Does she use her fingers to figure it out? Is this game appropriate for her?

## Your 7 Year Old

### COGNITIVE DEVELOPMENT

At seven children become more reflective. Emotionally, they may even seem moody and depressed. However, this moving inward, as seen in their choices of solo kinds of activities (like preferring to read alone in a corner), supports children's development of new cognitive structures and feelings.

A seven year old's movement between diligence and self-reflection helps create feelings of competence. Self-reliance plays a major role in learning. It is the foundation of autonomy and independence, critical tools for success in life. Developing this in early childhood sets the stage for the independence and self-direction needed at older ages.<sup>7</sup>

## MATH LEARNING AT AGE SEVEN

Because seven year old children have developed structures that enable them to understand space and quantity with increased sophistication, a broader range of mathematical ideas become more accessible. For many, earlier struggles with counting, conserving number, and part-whole relations have been resolved. Children now have a repertoire of basic addition and subtraction combinations that they can use as tools in computing. For example, to solve  $19 + 21$ , a child might think of a related combination,  $20 + 20$ . Some children may have constructed reversibility of thinking, a big developmental shift needed to understand how subtraction is the inverse of addition ( $50 - 25$  can be thought of as  $25 + ? = 50$ ).<sup>8</sup>

## ACTIVITIES

It is important that seven year olds be confident in their ability to reason mathematically. While a child may know a lot, he may not yet have developed ways to connect this knowledge. Try to find occasions where you might help him build connections. A question like, 'how much money is six quarters?' might be an opportunity for him to realize that if he knows that four quarters is one dollar, he could use that information to figure out what six quarters is. Be sure to highlight how important it is to gain confidence about what you know and how this self-knowledge is the starting place for problem solving.

Even though multiplication is on the horizon for many seven year olds, you might begin the journey by posing questions that help your student think in groups. For example, a question such as 'how many fingers do five people have?'

may be hard or easy for her to solve. Your challenge will be to find questions that support her reasoning.

## Your Eight Year Old

### COGNITIVE DEVELOPMENT

Broadly speaking, the early years can be characterized by times of great energy and eagerness to explore, followed by calmer times of reflection and practice; a year of rapid change and growth physically, socially, and cognitively, followed by a year to consolidate. Children's understanding and thinking gets steadily more complex.

The relatively solitary seven year old learner becomes the more social eight year old with perhaps a somewhat outsized view of himself and of what he can accomplish. The eight year old is often quick to start and quick to finish, with good ideas and good intentions.

The challenge for parents and teachers alike is how to help eight year olds develop their stamina for learning. Like any athlete, start small and get more challenging. Eight year olds can tire easily and will get satisfaction from finishing assignments if they are short ones. An individualized chart where an eight year old can see a record of her improvements may help her gain confidence in herself.

### MATH THINKING AT AGE EIGHT

At eight, children have developed some big ideas in mathematics, enabling them to use numbers and quantitative methods in more sophisticated ways. Eight year olds also have developed flexibility with number that includes "owning" a variety of strategies for computing in addition and subtraction. At this age children start to think more logically to solve problems and think about

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the world. The development of hierarchical classification is one example (e.g., eagles are birds, but not all birds are eagles. Since all birds are animals and eagles are birds, then eagles are also animals).

Cognitive shifts are happening in the world of number. Children now have developed reversibility of thinking, enabling them to understand subtraction as the inverse of addition. In the problem  $100 - 70$ , a child can think of the problem as a part/whole relation: if 100 is the whole and 70 is one part, then the missing part makes this equation true:  $70 + ? = 100$ .

A major developmental change occurs when moving into multiplication. This transition seems easy but it isn't, because multiplicative reasoning requires understanding number in a new way. In addition children may think of 130 as two or more addends that make a whole, but in multiplication the meaning of 130 is related to a unit that can shift. For example, if the unit is ten, 130 means 13 tens; however, if the unit is 100, 130 means 1.3 hundreds. This kind of reasoning (a big idea called unitizing) is the foundation for understanding the base-ten number system and future work with fractions, decimals, and percents.

### ACTIVITIES

Eight year olds can tend to take on too much and become frustrated when accomplishments don't come quickly or easily. It's important to support children as mathematical thinkers and help them develop confidence in their abilities. When a child is overwhelmed, help her break the task into many smaller tasks. First, identify what you know. Second, identify the problem to solve. Third, identify one way to try and solve the problem.

You can help develop your student's confidence by not just correcting a wrong answer. Instead try to follow his reasoning. Remember that answers come from the learner, from a logical certainty, and not from a teacher, parent, or textbook. Risk taking is a fundamental component of learning. By focusing on the process and not just on the answer, you will help him take risks and become an autonomous learner.

Since eight year olds are developing more complex ways of reasoning and using logic, they are drawn to games where everything has a reason and can be explained within the logic of the system. They especially love playing games where strategic thinking is employed (e.g., checkers, chess, Monopoly, Clue.)

Bear in mind that mathematical learning comes naturally through an emergent process, much like literacy acquisition. Helping children find enjoyment in problem solving will help them become life-long problem solvers.

### The Importance of Parental Participation in Learning

According to research conducted by Greg Duncan of Northwestern University, the single most important factor in predicting later academic success is that children begin school with a mastery of early math and literacy concepts. And math proficiency is most important. Mastery of early math predicts not only future math achievement, it also predicts future reading achievement.<sup>9</sup>

Parents are a child's first teacher — and research suggests that increased participation from parents can only enhance a child's ability to succeed. Regardless of socioeconomic status or ethnicity, studies show a strong correlation



between parental involvement and a child's academic achievement. In the early years especially, parental involvement has a significant impact on children's cognitive development and literacy and number skills.

Teachers can share this piece with parents to encourage involvement in learning at home, because the more parents know about what to expect at each stage of their child's growth, the better equipped they are with appropriate activities that will further their child's development. By fostering a positive environment for math learning at home through fun, everyday activities, parents can help their children become math literate and prepared for success in a changing world.

## Endnotes

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3. Cisneros, Sandra, short story, "Eleven," SAGUARO, vol. 3, University of Arizona, January 1, 1987
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7. Wood, Chip, Op. Cit.
8. Nunes, Terezinha and Bryant, Peter, "Children Doing Mathematics," Blackwell Publishing, 1996
9. You can read about the study at Northwestern University's website, [www.northwestern.edu/newscenter/stories/2007/11/duncan.html](http://www.northwestern.edu/newscenter/stories/2007/11/duncan.html). And you

Dynamic adaptations, based not just on answers but on strategies, keep all learners, from struggling to advanced, in their optimal learning zone.





Bellevue, Washington, and launched its first online learning product in January 2009. DreamBox Learning Math has won more than 35 top education and technology industry awards and is in use in all 50 U.S. states and throughout Canada. The DreamBox Learning Math platform offers a groundbreaking combination of Intelligent Adaptive Learning™ technology, a rigorous mathematics curriculum, and a highly motivating learning environment. DreamBox Learning Math captures every decision a student makes while working in the program and adjusts the student's learning path appropriately, providing millions of individualized learning paths, each one tailored to a student's unique needs. DreamBox supports teachers and their practice in every type of learning environment, offline or online. For more information about DreamBox Learning Math and the DreamBox Math for iPad app, please visit: [dreambox.com](http://dreambox.com).

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